Note to the reader. If a statement is marked with [Not proved in the lecture], then the statement was stated but not proved in the lecture. Of course, you don't need to know the proof.

If a statement is marked with [Proof not in the exam], then it was proved in the lecture, but the proof will not be asked in the exam.

If a statement is marked with [Statement not in the exam], then it will not be asked in the exam (both statement and proof).

Lecture 1:

Moral introduction to Topology

Lecture 2:

1. Definition: Open subset of $\mathbb{R}$.
2. Examples of open subsets: $\mathbb{R}, (a,b), \mathbb{R}-\{\text{ finite set }\}, \mathbb{R}-\left(\{\frac{1}{n}\}_{n=1}^{\infty} \cup \{0\}\right), \emptyset$.
3. A function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous (with the ε, δ definition) $\Leftrightarrow$ for each open set $O \subseteq \mathbb{R}$ the inverse image f^{-1} is open.
4. Definition: topological space, open set, topology, trivial topology, discrete topology.
5. The collection of all open subsets of $\mathbb{R}$ is a topology.
6. Definition: standard topology on $\mathbb{R}$.
7. Example: other topologies on $\mathbb{R}$: $\mathcal{T}_1 = \{\mathbb{R} - A \mid A \text{ finite }\} \cup \{\emptyset\}$; $\mathcal{T}_2 = \{O \subseteq \mathbb{R} \mid \forall x \in O \exists [a,b) \subseteq O \text{ with } x \in [a,b)\}$.
8. Definition: coarser topology, finer topology, closed set.
9. Let X be a set and let $\mathcal{C}$ be a family of subsetes of X such that
   (a) $\emptyset, X \in \mathcal{C}$;

(b) (finite union) $C_1, \ldots, C_n \in \mathcal{C} \Rightarrow \bigcup_{i=1}^n C_i \in \mathcal{C}$;

(c) (intersection) $\{C_i\}_{i \in I} \subseteq \mathcal{C} \Rightarrow \bigcap_{i \in I} C_i \in \mathcal{C}$.

Then $\{X - C \mid C \in \mathcal{C}\}$ is a topology on X.

Lecture 3:

1. Let X be a topological space and $A \subseteq X$. For every $x \in X$ exactly one of the following holds:
   (a) There exists O open with $x \in O \subseteq A$;
   (b) There exists O open with $x \in O \subseteq X - A$;
   (c) For all open sets O with $x \in O$, we have $O \cap A \neq \emptyset$ and $O \cap (X - A) \neq \emptyset$.
2. Definition: interior, boundary and closure of a subset $A \subseteq X$.
3. $\mathrm{Int}(A) \subseteq A \subseteq \overline{A}$.
4. Examples: $\overline{(a,b)} = [a,b]$, $\mathrm{Int}(\mathbb{Q}) = \emptyset, \partial\mathbb{Q} = \mathbb{R}$, when $\mathbb{Q}$ is seen as a subset of $\mathbb{R}$.
5. Let $A \subseteq X$ be a topological space. Then
   (a) $\mathrm{Int}(A)$ is open;
   (b) $\overline{A}$ is closed;
   (c) A is open $\Leftrightarrow A = \mathrm{Int}(A)$;
   (d) A is closed $\Leftrightarrow A = \overline{A}$.
6. Exercise:
   (a) $\mathrm{Int}(A)$ is the maximal open set contained in A;
   (b) $\overline{A}$ is the minimal closed set containing A.

Lecture 4:

1. Definition: Basis for a topology.
2. Remark: A basis $\mathcal{B}$ satisfies:
   (a) Every $x \in X$ lies in some $B \in \mathcal{B}$.
   (b) Let $B_1, B_2 \in \mathcal{B}$. If there is $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$ and $x \in B_3$.
3. If a family of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfies (a) and (b) as above, then it is a basis of a topology on X.
4. Definitions: Product topology on $X \times Y$, metric space, triangle inequality, ball of radius r around x (denoted $B_r(x)$).
5. Let (X, d) be a metric space. Then $\mathcal{B} = \{B_r(x) \mid x \in X, r > 0\}$ is a basis for a topology, called *metric topology*.
6. Examples of metric spaces: [Proof not in the exam]
   (a) X any set, $d(x, y) = \begin{cases} 0 & x = y, \\ 1 & x \neq y. \end{cases}$
   (b) $X = \{f \colon [0, 1] \to \mathbb{R} \mid f \text{ continuous}\}$, $d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$.
   (c) $X = \{f \colon [0, 1] \to \mathbb{R} \mid f \text{ continuous}\}$, $d_2(f, g) = \sqrt{\int_0^1 |f(x) - g(x)|^2 dx}$.
7. Definition: subspace topology.
8. Let $A \subseteq X$ be open (resp. closed). Then $B \subseteq A$ is open (resp. closed) in the subspace topology $\Leftrightarrow$ it is open (resp. closed) as a subset of X.
9. A function $f \colon X \to Y$ is continuous $\Leftrightarrow$ for all $C \subseteq Y$ closed, we have that $f^{-1}(C)$ is closed in X.
10. Let $f \colon X \to Y$ be a function.

(a) Given a basis $\mathcal{B}$ for Y, f is continuous $\Leftrightarrow$ $f^{-1}(B)$ is open for all $B \in \mathcal{B}$.

(b) $f\colon X \to Y$, $g\colon Y \to Z$ continuous $\Rightarrow$ $g \circ f\colon X \to Z$ is continuous.

(c) $f\colon X \to Y$ continuous, $A \subseteq X$. Then $f_{|A}$ is continuous with respect to the subspace topology on A.

(d) Let $f\colon X \to Y$, $g\colon W \to Z$. Then $f \times g$ is continuous $\Leftrightarrow$ f and g are.

11. Definition: Homeomorphism.

12. Let X be a set of cardinality at least 2. Then the identity map $\mathrm{Id}\colon (X, \mathcal{T}_{\text{discrete}}) \to (X, \mathcal{T}_{\text{trivial}})$ is continuous but not a homeomorphism.

Lecture 5:

1. A topological space X is *disconnected* if one of the following equivalent conditions hold:

   (a) X is the union of disjoint, non-empty open sets;

   (b) X is the union of disjoint, non-empty closed sets;

   (c) There exists $A \subset X$, where $X \neq A \neq \emptyset$, which is open and closed.

2. Definition: connected space.

3. Examples: $\mathbb{R} - \{0\}$ is disconnected, $\mathbb{Q}$ is disconnected, the topology of $\mathbb{R}$ given by the basis $\{[a, b)\}$ is disconnected.

4. $[a, b]$ is connected.

5. Definition: path connected space.

6. If X is path-connected, then it is connected.

7. Let $f\colon X \to Y$ be a continuous surjective map and suppose that X is (path-)connected. Then Y is (path-)connected.

Lecture 6:

1. Definition: cut-point
2. The number of cut-points is invariant under homeomorphism.
3. Let $Z = \{(x, \sin\left(\frac{1}{x}\right) \mid x > 0\} \cup \{(0, x) \mid x \in [-1, 1]\}$. Then Z is connected but not path-connected. [Proof not in the exam]
4. Let $P(x) = \{y \in X \mid$ there exists a path from x to $y\}$ (called *path-connected component of* x). If $P(x) \cap P(y) \neq \emptyset$, then $P(x) = P(y)$.
5. Definition: inverse path, connected/disconnected/path-connected subspace.
6. A topological space is the disjoint union of its path-connected components.
7. If $A \subseteq X$ is a path-connected subspace, then it is contained in a path connected component of X.
8. if $f\colon X \to Y$ is a continuous function , then $f(P(x)) \subseteq P(f(x))$.
9. If $A \subseteq X$ is a connected subspace, then $\overline{A}$ is connected.
10. Remark: this is not true with $\mathrm{Int}(A)$.
11. Let $A \subseteq X$ be closed and open, and $C \subseteq X$ be connected. Then if $A \cap C \neq \emptyset$, then $C \subseteq A$.
12. Let $\{C_\alpha\}$ be a family of connected subspaces of X. If $C_\alpha \cap C_\beta \neq \emptyset$ for all α and β, then $\bigcup C_\alpha$ is connected.
13. Definition: connected component $C(x)$ of a point x.

14. $C(x)$ is connected, closed and if $C(x) \cap C(y) \neq \emptyset$, then $C(x) = C(y)$.

15. Definition: neighbourhood, locally path connected topological space.

16. Suppose that X is locally path connected. Then for each $x \in X$, $P(x) = C(x)$.

17. As a corollary we get that if X is locally path connected, then X is connected $\Leftrightarrow$ X is path-connected.

Lecture 7:

1. Definitions: totally disconnected space, isolated point, Cantor Set, bounded set.

2. Let $C \subseteq \mathbb{R}$ be the Cantor set. Then C is non-empty, totally disconnected, closed, without isolated points.

3. Let $X \subseteq \mathbb{R}$ be a non-empty, totally disconnected, closed, bounded subset without isolated points. Then X is homeomorphic to the Cantor set. [Not proved in the lecture]

Lecture 8:

1. Definitions: Compact, open cover, subcover.

2. Example: $\mathbb{R}$ is not compact.

3. Let X be a compact, non empty, topological space, $f\colon X \to \mathbb{R}$ a continuous function. Then f has a maximum.

4. The interval $[a, b]$ is compact in $\mathbb{R}$.

5. Let X a compact set and $Y \subseteq X$ be a closed set. Then Y is compact.

6. Let X be a topological space and $\mathcal{B}$ be a basis for the topology. Then X is compact if and only if for every open cover $\{O_\alpha\}$ such that $O_\alpha \in \mathcal{B}$ for all α, there exists a finite subcover.

7. Let $f\colon X \to Y$ be continuous and surjective, and assume that X is compact. Then Y is compact.

8. Let X and Y be compact topological spaces. Then $X \times Y$ is compact.

9. A subset X of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Lecture 9:

1. Want a topology for infinite products $X = \prod_{i\in I} X_i$. First candidate: choose the basis $\mathcal{B} = \{O_1 \times \cdots \times O_n \times \cdots \mid O_i \text{ open in } X_i\}$. However, for $X = \mathbb{R}^{\mathbb{N}}$ the basis $\mathcal{B}$ would make the map $f\colon \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ defined as $f(x) = (x, x, x, \dots)$ not continuous.

2. Definition: product topology (for arbitrary products).

3. Let Z and X_i, for $i \in I$, be topological spaces. For each $i \in I$, let $f_i\colon Z \to X_i$ be a function. Then the function $f\colon Z \to \prod_{i\in I} X_i$ defined as $f(z) = (f_1(z), \dots, f_n(z), \dots)$ is continuous if and only if each f_i is continuous (the product $\prod_{i\in I} X_i$ is equipped with the product topology).

4. Let X_i, for $i \in I$ be topological spaces, and let $X = \prod_{i\in I} X_i$ be equipped with the product topology. Then X is compact if and only if each X_i is compact.[Proof not in the exam]

Lecture 10:

1. Definition:Limit of a sequence, first countable space, neighbourhood basis, Hausdorff space, converging sequence.

2. Non-example: Zariski topology.

3. Let X be a Hausdorff topological space. Then any sequence has at most one limit.

4. Let X be a metric space with the metric topology. Then X is first countable and Hausdorff.

5. Let X be a first countable metric space. Then for every subset A of X:

   (a) $\overline{A} = \{x \mid \exists \{x_n\}$ with each $x_n \in A$, such that $x = \lim\{x_n\}\}$.

   (b) A is closed if and only if the limit of a sequence contained in A is contained in A (in case such a limit exists).

6. Let X, Y be topological spaces and assume that X is first countable. Then $f\colon X \to Y$ is continuous if and only if for every sequence $\{x_n\}$ in X with limit point x, $f(x)$ is the limit of $\{f(x_n)\}$.

7. Definition: sequentially compact space, totally bounded space, second countable space.

8. Let X be a metric space. Then the following are equivalent:

   (a) X is compact;

   (b) X is sequentially compact;

   (c) X is complete and totally bounded.

9. Let X be a first countable topological space that is compact. Then X is sequentially compact. (The proof is part of the statement before)

10. Let X be a second countable topological space that is sequentially compact. Then X is compact. (The proof is part of the statement before)

Lecture 11:

1. Definition: Lebesgue number, for a subset A of a metric space X, $d(x, A) = \inf_{y \in A}\{d(x, y)\}$.
2. Let X be a compact metric space. Then every open cover has a Lebesgue number.
3. For a subset A of a metric space X and a point $x \in X$ it holds
   (a) $d(x, A) \geq 0$ and $d(x, A) = 0 \iff x \in \overline{A}$.
   (b) $d(x, A)$ is 1-Lipschitz (hence continuous) in x.
4. Definition: uniformly continuous function.
5. Let X, Y be compact metric spaces. Then every continuous function $f\colon X \to Y$ is uniformly continuous.

Lecture 12:

1. Definition: Space $C(X, Y)$, equicontinuous family of functions.
2. [Proof not in the exam]Let X and Y be compact metric spaces. Then a set $\mathcal{F} \subseteq C(X, Y)$ is compact if and only if it is closed and equicontinuous.
3. [Statement not in the exam]Let $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous bounded function, and let $t_0 \in \mathbb{R}$. Then the system
$$\begin{cases} y'(x) = f(x, y(x)) \\ y(0) = t_0 \end{cases}$$
   has a solution $y\colon [0,1] \to \mathbb{R}$.
4. Definition: dense subset.
5. [Statement not in the exam]There exists $f\colon [0,1] \to \mathbb{R}$ continuous such that $f'(x)$ does not exist for all $x \in [0,1]$.

Lecture 13:

1. Definition: one-point compactification.

2. Let X be a Hausdorff space and C be a compact subspace of X. Then C is closed in X.

3. Let X be a Hausdorff space and C be a compact subspace of X. Then for every $x \in X - C$ there are disjoint open sets U and V such that $x \in U$ and $C \subseteq V$.

4. Let X be a non-compact Hausdorff space and $\widehat{X}$ be the one-point compactification of X. Then $\widehat{X}$ is compact and X is dense in $\widehat{X}$.

Lecture 14:

1. Example: stereographic projection: $\widehat{\mathbb{R}^n}$ is homeomorphic to S^n [Proof not in the exam]

2. Definitions: normal space, regular space.

3. Let X be a compact space and Y be a hausdorff space, and let $f : X \to Y$ be a continuous map. Then f is closed (i.e. for each closed set $C \subseteq X$, $f(C)$ is closed in Y).

4. [Corollary] A continuous bijection between a compact space and a hausdorff space is a homeomorphism.

5. Let X be a non-compact Hausdorff space. Then $\widehat{X}$ is Hausdorff if and only if X is locally compact. [Proof not in the exam]

6. Let X a locally compact Hausdorff space, and let $\{U_n\}$ be a family of open dense subsets. Then $\bigcap U_n$ is dense. [Proof not in the exam]

7. Let $(X, \mathcal{T})$ be a second-countable Hausdorff space. Then there exists a distance on X that induces the topology $\mathcal{T}$ [Proof not in the exam].

Lecture 15:

1. Let X be a compact Hausdorff space. Then X is normal (in particular is regular).

2. Let X be a metric space. Then X is normal.

3. Definition: Quotient topology, quotient space.

4. Informal discussion: quotient = gluing.

5. Let X be a compact space, Y be a hausdorff space and let $f: X \to Y$ be a continuous surjection. Then a set $U \subseteq Y$ is open in Y if and only if $f^{-1}(U)$ is open in X.

Lecture 16:

1. Definition: equivalence relation, equivalence class, quotient by an equivalence relation.

2. Let X, Y be topological spaces and let $f: X \to Y$ be a quotient map. Let $\sim$ be the equivalence relation on X defined as $x \sim y$ if $f(x) = f(y)$. Then the map $\overline{f}: X/_{\sim} \to Y$ defined as

$$[x] \mapsto f(x)$$

is well defined and a homeomorphism.

3. Let X, Y be topological spaces and let $\sim$ be an equivalence relation on X. Let $f: X \to Y$ be a continuous map such that whenever $x \sim y$ we have $f(x) = f(y)$. Then the map $\overline{f}: X/_{\sim} \to Y$ defined as $\overline{f}([x])$ is well defined and continuous.

4. [Statement not in the exam]Examples: $[0,1], 0 \sim 1$ is homeomorphic to S^1, $X = \mathbb{R} \times \{0,1\}$, $(x,0) \sim (x,1)$ if $x \neq 0$ (*line with two zeroes*). Quotient is not hausdorff, contains non-closed compact subsets, exists compact subsets whose intersection is not compact.

5. Let X be a topological space, let A be a dense subset of X and let Y be a Hausdorff space. Suppose that there are continuous functions $f_1, f_2 \colon X \to Y$ such that for every $a \in A$ we have $f_1(a) = f_2(a)$. Then $f_1 = f_2$.

6. [Statement not in the exam]Let X be a Hausdorff, path connected topological space. Then X is *arc connected* (i.e. any two points can be joined by an injective arc).

7. Examples: Let $Q = [0,1] \times [0,1]$. Then the cylinder is given by $(0,1) \sim (1,t)$, for all $t \in [0,1]$. The Möbius strip is given by $(0,t) \sim (1, 1-t)$. The torus is given by $(0,t) \sim (1,t)$ and $(s,0) \sim (s,1)$. The torus is homeomorphic to $S^1 \times S^1$.

Lecture 17:

1. The sphere S^2 can be described as a quotient in several ways. Presented the following:

   (a) $X = D^2 \times \{a,b\}$, where D^2 is the unit disk. Then $S^2 \cong X/_\sim$, where $(v,a) \sim (v_b)$ for each $v \in S^1$ (note that S^1 is hte boundary of D^2).

   (b) $X = D^2$ and $S^2 \cong X/_\sim$ where $(x,y) \sim (x,-y)$ for each $(x,y) \in S^1$.

   (c) $X = D^2$ and $S^2 \cong X/S^1$.

2. Definition of Klein bottle, projective plane.

3. The projective plane is homeomorphic to $S^2/_\sim$, where $x \sim -x$ for all x.

Lecture 18:

1. Let X be a normal topological space and $q\colon X \to Y$ be a closed quotient map. Then Y is normal.[Proof not in the exam]

2. Let X be a compact Hausdorff space, let $\sim$ be an equivalence relation on X and assume that $R = \{(x, y) \mid x \sim y\} \subseteq X \times X$ is closed. Then the projection map $q\colon X \to X/_\sim$ is closed. [Proof not in the exam]

3. [Statement not in the exam]There is an equivalence relation on the Cantor set C such that $C/_\sim$ is homeomorphic to $[0, 1]$.

4. [Statement not in the exam]Let X be a compact metric space. Then X can be realized as a quotient of the Cantor set.

5. [Statement not in the exam]Definition of Sierpinsky carpet.

6. [Statement not in the exam]There is an equivalence relation on the Sierpinsky carpet S such that $S/_2$ is homeomorphic to the sphere S^2.

Lecture 19:

1. What is a topological invariant;

2. Definitions: loop, homotopy (of paths), linear homotopy;

3. Being homotopic is an equivalence relation.

Lecture 20:

1. Definitions: null-homotopic loop, fundamental group,

2. The fundamental group is well defined. Its trivial element is the constant loop and the inverse of a class $[\alpha]$ is the class $[\alpha^{-1}]$, where $[\alpha^{-1}]$ is the path obtained reversing the orientation of α.

3. Let X, Y be topological spaces. Assume that $X = A \cup B$, with A, B closed, and let $f\colon X \to Y$. If $f_{|A}$ and $f_{|B}$ are continuous, so it is f. This is still true if instead of considering 2 closed sets, we consider finitely many.

4. Remark: The above is still true if both A and B are open or, more general, if X is the union of arbitrarily many open. It is not true for general A and B.

5. Let X be a topological space, and let γ be a path between two points x_0 and x_1. Then there exists an isomorphism $\beta_\gamma\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$, where $\beta_\gamma([\alpha])$ is defined as $[\gamma^{-1} * \alpha * \gamma]$.

6. Corollary: if X is path connected, then for every pair x_0, x_1 we have that $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. In this case we can denote $\pi_1(X, x_0)$ simply as $\pi_1(X)$.

7. Remark: the isomorphism β_γ depends on the choice of γ.

8. [Peano Curve] There exists a continuous surjective map $[0, 1] \to [0, 1] \times [0, 1]$.[Statement not in the exam]

Lecture 21:

1. Let X be a topological space and suppose that X can be written as $X = \bigcup U_\alpha$, where each U_α is a path-connected open subset of X. Suppose, moreover, that there exists x_0 such that $x_0 \in U_\alpha$, for every α and that $U_\alpha \cap U_\beta$ is path-connected for every α and β. Then every loop at x_0 is homotopic to a concatenations of loops such that every loop in the concatenation is contained is some U_α.

2. As a corollary, we obtain that $\pi_1(S^n) = \{1\}$, for $n > 1$.

3. [Proof next lecture] $\pi_1(S^1) \cong \mathbb{Z}$.

4. [Brouwer fixed point Theorem] Let $f\colon D^2 \to D^2$ be any continuous map. Then f has a fixed point, that is, there exists $x \in D^2$ such that $f(x) = x$.[Statement not in the exam]

Lecture 22:

1. The fundamental group of the circle S^1 is isomorphic to the integers, and it is generated by the class of the loop $w(t) = (\cos(2\pi t), \sin(2\pi t))$.
2. Definition: Covering, evenly covered neighbourhood, covering space, covering map, lift(ing) of a map f,
3. Example: the quotient map $S^2 \to \mathbb{RP}^2$ is a covering.
4. Let $p\colon \widehat{X} \to X$ be a cover, let $\widehat{x_0} \in \widehat{X}$ be a point, and let $x_0 = p(\widehat{x_0})$. Then we have:
   (a) [Unique lift of paths] For every path $f\colon [0,1] \to X$ with $f(0) = x_0$, there exists a unique lift $\widehat{f}$ with $f(\widehat{x_0}) = x_0$.
   (b) [Unique lift of homotopies] For every homotopy $F\colon [0,1] \times [0,1] \to X$ with $F(0,0) = x_0$, there exists a unique homotopy $\widehat{F}\colon [0,1] \times [0,1] \to \widehat{X}$ with $\widehat{F}(0,0) = \widehat{x_0}$.
5. Let $p\colon \widehat{X} \to X$ be a cover, assume that $\widehat{X}$ is path connected, and let $\widehat{x_0} \in \widehat{X}$ and $x_0 = p(\widehat{x_0})$. Let $\sigma\colon \pi_1(X, x_0) \to p^{-1}(x_0)$ be the map defined as $\sigma([\gamma]) = \widehat{\gamma}(1)$, where $\widehat{\gamma}$ is the unique lift of γ such that $\widehat{\gamma}(0) = \widehat{x_0}$. Then σ is well-defined and surjective. Moreover, if $\pi_1(\widehat{X}, \widehat{x_0}) = \{1\}$, it is also injective.
6. As a corollary we obtain $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$.

Lecture 23:

1. [Proof as exercise] Let X, Y be topological spaces and let $f\colon X \to Y$ be a continuous map. Assume that $f(x_0) = y_0$. Then the

map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined as $f_*([\gamma]) = [f \circ \gamma]$ is well-defined and a group homeomorphism.

2. As a corollary we obtain that if X and Y are path-connected and homeomorphic, then $\pi_1(X) \cong \pi_1(Y)$.

3. Definition: homotopy between continuous maps (from general topological spaces), homotopy equivalence between topological spaces.

4. Being homotopic equivalent is an equivalence relation between topological spaces.

5. Examples: $\mathbb{R}^n$ is homotopic equivalent to a point, the Möbius strip is homotopic equivalent to S^1.

Lecture 24:

1. $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for any $n > 2$. (Note that we already know that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$).

2. We have: $\mathbb{R}^k - \{0\} \simeq S^{k-1}$, where $\simeq$ denotes homotopy equivalence.

3. Let $f, g\colon X \to Y$ be two homotopic maps, and let $x_0 \in X$. Then there is a path γ in Y from $f(x_0)$ to $g(x_0)$ such that $\beta_\gamma \circ g_* = f_*$ (recall that $\beta_\gamma([\alpha]) = [\gamma * \alpha * \gamma^{-1}]$). This amounts to say that there exists γ such that the following diagram commutes.

$$\begin{tikzcd}
& \pi_1(Y, f(x_0)) \arrow[dd, "\beta_\gamma"] \\
\pi_1(X, x_0) \arrow[ur, "g_*"] \arrow[dr, "f_*"'] & \\
& \pi_1(Y, g(x_0))
\end{tikzcd}$$

4. As a corollary we obtain that if $f\colon X\to Y$ is a homotopy equivalence and $x_0\in X$, then $f_*\colon \pi_1(X,x_0)\to\pi_1(Y,f(x_0))$ is an isomorphism.

Lecture 25:

1. Definition: free group, reduced word, free product of groups.

2. [Proof not in the exam]The free product $*G_a$, is a group, and each G_a is (naturally identified with) a subgroup.

3. Definition: free group.

4. [Proof not in the exam]Let $\{G_a\}$ be a family of groups, and H be a group. For each a, let $\psi_a\colon G_a\to H$ be a homomorphism. Then there exists a unique map $\psi\colon *G_a\to H$ such that $\psi_{|G_a}=\psi_a$.

5. [Proof not in the exam]Let G be a group and $A\subseteq G$. Then there exist a unique minimal normal subgroup (with respect to inclusion) containing A, denoted by $\langle\langle A\rangle\rangle$. Moreover

$$\langle\langle A\rangle\rangle=\left\{\prod_{i=1}^{n} g_i a_i^{\pm} g_i^{-1}\mid g_i\in G, a_i\in A\right\}.$$

6. Definition: presentation of a group (denoted $\langle S\mid R\rangle$), relators.

7. [Proof not in the exam]Let $G=\langle S\mid R\rangle$, $N=\langle\langle R\rangle\rangle\lhd F_S$ and let $\phi\colon F_S\to H$ be a homomorphism. If for all $r\in R$ we have that $f(r)=\mathrm{Id}$, then $\overline{\phi}\colon G\to H$ given by $\overline{\phi}(gN)=\phi(g)$ is a well-defined homomorphism.

8. [Van Kampen Theorem] [Proof not in the exam]Let X be a topological space, $A,B\subseteq X$ be open path-connected, and let

$x_0 \in A \cap B$. Let

$$\begin{aligned} i_A \colon A &\to X \\ i_B \colon B &\to X \\ i_{A,B} \colon A \cap B &\to B \\ i_{B,A} \colon A \cap B &\to A \end{aligned}$$

be the inclusions. Let

$$\phi = \phi(\{(i_A)_*, (i_B)_*\}) \colon \pi_1(A, x_0) * \pi_1(B, x_0) \to \pi_1(X, x_0).$$

Then ϕ is surjective and the Kernel of ϕ is

$$\langle\langle \big\{ (i_{A,B})_*(g) \cdot (i_{B,A})_*(g^{-1}) \mid g \in \pi_1(A \cap B, x_0) \big\} \rangle\rangle.$$

9. [Corollary] If A and B are simply connected, then so it is X.
10. $\pi_1(S^2) \cong \{1\}$.
11. Definition: Rose R_n.
12. $\pi_1(R_n) \cong F_n$, where F_n is the free group on n generators.

Lecture 26:

[Statement not in the exam] Proof of Van Kampen Theorem

Lecture 27:

[Statement not in the exam] Covering Theory.